

**Objectives:**

- Take derivatives of polynomials using the power rule, the constant multiple rule, and the sum/difference rule.

**Derivative of a constant:**

$$\frac{d}{dx}(c) = 0$$

*Proof:*

$$\begin{aligned} f(x) &= c \\ \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{c - c}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = \lim_{h \rightarrow 0} 0 = 0 \end{aligned}$$

**Derivative of  $f(x) = x$ :**

$$\frac{d}{dx}(x) = 1$$

*Proof:*

$$\begin{aligned} f(x) &= x \\ \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{x+h-x}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = \lim_{h \rightarrow 0} 1 = 1 \end{aligned}$$

**Derivative of  $f(x) = x^2$ :**

$$\frac{d}{dx}(x^2) = 2x$$

*Proof:*

$$\begin{aligned} f(x) &= x^2 \\ \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(2x+h)}{h} = \lim_{h \rightarrow 0} (2x+h) = 2x \end{aligned}$$

Derivative of  $f(x) = x^3$ :

$$\frac{d}{dx}(x^3) = 3x^2$$

*Proof:*

$$\begin{aligned} f(x) &= x^3 \\ \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(3x^2 + 3xh + h^2)}{h} \\ &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2) \\ &= 3x^2 \end{aligned}$$

**Power Rule:**

For any nonzero number  $n$ ,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

*Proof:* (for positive integers  $n$ )

$$\begin{aligned} f(x) &= x^n \\ \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n - x^n}{h} \\ &= \lim_{h \rightarrow 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h} \\ &= \lim_{h \rightarrow 0} \frac{h \left( nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1} \right)}{h} \\ &= \lim_{h \rightarrow 0} \left( nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1} \right) \\ &= nx^{n-1} \end{aligned}$$

**Constant Multiple Rule:**

$$\frac{d}{dx}(cf(x)) = c \frac{d}{dx}(f(x))$$

*Proof:*

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{cf(x+h) - cf(x)}{h} &= \lim_{h \rightarrow 0} c \left( \frac{f(x+h) - f(x)}{h} \right) \\ &= c \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= c \frac{d}{dx}(f(x)) \end{aligned}$$

**Sum/Difference Rule:**

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x)) \quad \text{and} \quad \frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x))$$

*Proof:* (of the sum rule)

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h} &= \lim_{h \rightarrow 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h} \\ &= \lim_{h \rightarrow 0} \left( \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right) \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ &= \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x)) \end{aligned}$$

Some practice with our new rules!

1. Find the derivatives of the following functions using our new rules:

(a)  $f(x) = 3x^5$

$$\frac{d}{dx}(3x^5) = 3 \frac{d}{dx}(x^5) = 3 \cdot 5x^4 = 15x^4$$

(b)  $f(x) = \frac{4}{\sqrt[3]{x}} + 3x = 4x^{-1/3} + 3x$

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(4x^{-1/3} + 3x) = 4 \frac{d}{dx}(x^{-1/3}) + 3 \frac{d}{dx}(x) = 4 \left( -\frac{1}{3}x^{-4/3} \right) + 3 \cdot 1 = -\frac{4}{3}x^{-4/3} + 3$$

(c)  $f(x) = \frac{x^3 + \sqrt{x}}{x} = x^2 + x^{-1/2}$

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(x^2 + x^{-1/2}) = 2x - \frac{1}{2}x^{-3/2}$$

(d)  $f(t) = \sqrt[5]{t} + \frac{3}{t^2} + \frac{1}{4t} = t^{1/5} + 3t^{-2} + \frac{1}{4}t^{-1}$

$$\frac{d}{dt}(f(t)) = \frac{d}{dt} \left( t^{1/5} + 3t^{-2} + \frac{1}{4}t^{-1} \right) = \frac{1}{5}t^{-4/5} - 6t^{-3} - \frac{1}{4}t^{-2}$$

(e)  $g(s) = 4s^{10} + 3s^4 + 7$

$$\frac{d}{ds}(4s^{10} + 3s^4 + 7) = 40s^9 + 12s^3$$

(f)  $h(x) = \left( \sqrt[3]{x} + \frac{4}{\sqrt[3]{x}} \right)^2 = (x^{1/3} + 4x^{-1/3})^2 = x^{2/3} + 8 + 16x^{-2/3}$

$$\frac{d}{dx}(h(x)) = \frac{d}{dx}(x^{2/3} + 8 + 16x^{-2/3}) = \frac{2}{3}x^{-1/3} - \frac{32}{3}x^{-5/3}$$

2. Find an equation of the tangent line to  $f(x) = 3x^4 + x^2 + 1$  at  $x = 1$ .

$$f'(x) = 12x^3 + 2x \quad f'(1) = 14$$

The point  $(1, 5)$  lies on the curve and the line tangent to the curve at  $x = 1$  so the equation of our tangent line is

$$y - 5 = 14(x - 1)$$